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Continued Fractions

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Continued Fractions

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Report

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Abstract

Continued Fractions

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This report examines the theory of continued fractions and how their use enhances the secondary mathematics curriculum. The use of continued fractions to calculate best approximants of real numbers is justified geometrically, and famous irrational numbers are described as continued fractions. Periodic continued fractions and other applications of continued fractions are also discussed.

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Chapter 1: An Introduction

Continued fractions are used to model and study objects as small as DNA, as large as galaxies, and an extensive list of things in between. Their study has implications for number theory, computer applications, and even biology. Simple continued fractions are so basic in nature that the author has been able to successfully teach sixth graders to correctly calculate the partial quotients of any rational number. Nevertheless, the theory of continued fractions is largely overlooked, today, in the secondary and even in the tertiary classroom. Often, the first encounter a student will have with continued fractions comes while taking a class on number theory and such a class is not always a requirement to earn a degree in mathematics.

It is not the goal of this paper to argue that the study of continued fractions be added to an already overstuffed public school curriculum. However, it is commonly acknowledged that most students dislike fractions. Some have even argued for the complete abandonment of fractions as a topic of study. Proponents of this position argue that decimals are enough and that fractions are so complicated that they turn students off to mathematics at an early age. While the author rejects this position entirely, it is clear that students are not being sufficiently motivated to study fractions. Perhaps, if students were exposed to some of the richness and applications of continued fractions, they would develop a greater interest in common fractions.

With this goal in mind, this paper discusses the theory behind continued fractions. The usage of continued fractions to calculate best approximants of real numbers is justified geometrically. Various techniques for calculating partial quotients for the continued fraction forms of π (pi), e, and φ (the golden ratio) will then be examined. Then, the connection between periodic continued fractions and quadratic surds will be discussed with focus placed on the impact of this connection on number theory. Finally, suggestions are given for the inclusion of continued fractions in the secondary mathematics curriculum as enrichment tools to motivate the study of proper fractions.

Chapter 2: The Simple Continued Fraction as the Best Approximant

Simple continued fractions are fractions of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where the a_i are all positive integers when $i > 0$, and a_0 can be a positive or negative integer, or zero. The a_i are called partial quotients and it is common practice to list them to designate the continued fraction with the notation

$$[a_0; a_1, a_2, a_3, a_4, \cdots]. \tag{1}$$

If the continued fraction has a final member a_n, then it is said to be finite. If there is no final partial quotient, then the sequence above gives an infinite continued fraction [**4**, p. 130].

Taking any real number β, designate the integral part of β as $[\ \beta\ ]$ and the fractional part of β as $\{\ \beta\ \}$. Then, define

$$\beta_0 = [\ \beta\ ]$$

and

$$\beta_n = \frac{1}{\{\beta_{n-1}\}}$$

where

$$n \geq 1.$$

Finally, define $a_i = \beta_i$ to give the continued fraction in (1).

If the continued fraction is finite and N is an integer, then

$$[a_0; a_1, a_2, a_3, a_4, \ \cdots \ , a_N] = \frac{p_N}{q_N} = \beta,$$

and β is a rational number. If the continued fraction is infinite, then β is irrational and the rational number

$$[a_0; a_1, a_2, a_3, a_4, \ \cdots \ , a_n] = \frac{p_n}{q_n}$$

is called the nth order approximant of β (**8,** p. 697).

The value of this nth approximant of β is that it is the best rational approximation of β. A fraction $\frac{p}{q}$, where $q \neq 0$, is called the best rational approximation of the irrational number β if for all fractions $\frac{p'}{q'}$

$$|q\beta - p| < |q'\beta - p'|$$

with

$$0 < q' \le q$$

unless

$$q = q' \text{ and } p = p'. \qquad (\mathbf{8}\text{, p. 697})$$

One would expect that the above definition would use the distance,

$$\left|\beta - \frac{p}{q}\right|.$$

Indeed, when comparing only fractions on one side of β the distance is sufficient. Instead, by multiplying both sides of the inequality by the denominator (q or q') fractions from either side of β may be compared. Richards provides detail on why

$$|q\beta - p|,$$

which he calls the "ultra-distance", allows for such comparison in defining the best rational approximation (**3,** p. 167).

Several famous mathematicians have provided proofs that the simple continued fraction provides the best rational approximation of any irrational number. It can be argued that the first one to demonstrate this aspect of continued fractions was Euclid, since methods he pioneered are directly employed in its traditional proofs (**5**, p. 368). Euler, Lagrange, and Gauss laid the foundations of how continued fractions are currently understood, and each of these men referred to how the continued fraction of an irrational number can be used to find the best rational approximations of the irrational (**10**, p. 203). These earlier writings emphasized algebraic methods in their proofs. More recently, Ford (**7**), Richards (**3**) and Irwin (**8**) have published geometric arguments.

The author has a particular fondness for Irwin's treatment because he believes that it can be informally transferred to the typical high school student or to the gifted student in middle school. First, construct an integer lattice $\mathbb{Z}^2$. Each point on this lattice represents a fraction with the ordinate as the numerator and the abscissa as the denominator. For example, in Figure 1 the lattice point A has coordinates (3, 2) and will represent $\frac{2}{3}$.

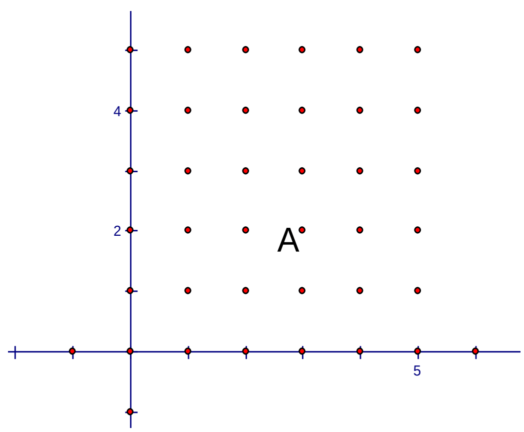

Figure 1. The $\mathbb{Z}^2$ Lattice.

Let l be the line

$$y = \beta x.$$

Then, in order for $\frac{p_n}{q_n}$ to be the best approximation of β, of all the points (q', p') on the lattice where

$$0 < q' < q_n,$$

(q_n, p_n) will be uniquely the nearest lattice point to l when nearness is measured as the vertical distance from (q', p') to l (**8,** p. 699).

In terms of vectors, let **O** be the origin and let **A** and **B** be integer lattice points. Then, **F** is the vector **A + B** and **P** is the intersection of the line l with $\overline{BF}$ (see Figure 2). The vector point **C** is called the "outpoint" of **A** and **B** and is denoted

$$C = C(A, B; \beta).$$

Q is the point of intersection of l and the extension of $\overline{AF}$. It follows that

$$P = \vartheta A + B$$

where

$$0 < \vartheta < 1,$$

and

$$Q = A + \frac{1}{\vartheta}\beta.$$

Therefore, Q lies between

$$C = A + \left[\frac{1}{\vartheta}\right]\beta$$

and

$$B + C.$$

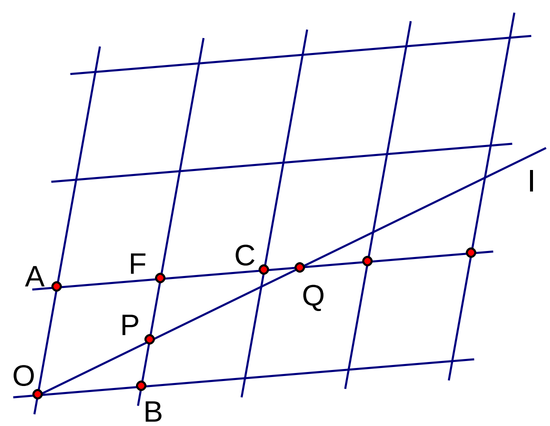

Figure 2. Constructing the Outpoint.

It is clear that of all the lattice points with abscissa less than or equal to the abscissa of **C**, **C** is the closest lattice point to *l*. Iterating this construction, let

$$C_2 = C(B, C; \beta),$$
$$C_3 = C(C, C_2; \beta),$$

and

$$C_i = C(C_{i-2}, C_{i-1}; \beta)$$

where i is an integer and

$$3 \le i \le n.$$

It has been shown that each C_i denotes a best rational approximation of β. It remains to be demonstrated that $\frac{p_n}{q_n}$ yields the same. Letting l' denote the line

$$y = \frac{p_n}{q_n} x,$$

Irwin shows that the outpoint constructions yield the exact same points C_i for l' as they do for l. Letting d'_i be the vertical distance from C_i to l' it follows that if

$$\frac{d'_i}{d'_{i-1}} = \vartheta'_i,$$

then

$$\frac{d'_{i-2}}{d'_{i-1}} = \frac{1}{\vartheta'_{i-1}} = a_i + \frac{d'_i}{d'_{i-1}}.$$

Continuing this substitution yields that

$$\left\{\frac{p_n}{q_n}\right\} = [0; a_1, \cdots, a_n],$$

and thus

$$\left[\frac{p_n}{q_n}\right] = [a_0; a_1, \cdots, a_n]. \qquad (\mathbf{8}, \text{p. } 701)$$

Finally, since each $\frac{p_n}{q_n}$ corresponds to an outpoint which represents a best rational approximation, it is directly shown that the simple continued fraction will always yield a best rational approximation for any irrational number when using the first n partial quotients.

Chapter 3: Famous Examples Calculated

In the previous chapter, it was shown that truncated continued fractions generate the best rational approximants of any irrational number. This chapter explores how to calculate the partial quotients of any real number with special emphasis placed on pi, *e*, and the golden ratio. Finally, it will be shown how to use matrices to simplify any terminating continued fraction.

The continued fraction for pi is

$$\pi = [3;\ 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \cdots]. \qquad (\mathbf{6}, \text{p. } 456)$$

To calculate these partial quotients for pi, start with an accepted decimal approximation for pi:

$$\pi = 3.141592654.$$

Truncating the whole portion of pi, set the first partial quotient:

$$a_0 = 3.$$

Set the whole portion of reciprocal of the decimal portion (0.141592654) as the second partial quotient:

$$a_1 = \left[\frac{1}{0.141592654}\right] = [7.062513306] = 7.$$

Next, repeat the previous step, setting the whole portion of the reciprocal of the leftover decimal portion (0.062513306) as the third partial quotient:

$$a_2 = \left[\frac{1}{0.062513306}\right] = [15.99659441] = 15.$$

Repeating this process leads to

$$\pi = [3;\ 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \cdots]. \qquad (\mathbf{6}, \text{p. } 456)$$

The same process can be used to calculate the following results:

$$e = [2;\ 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \cdots]. \qquad (\mathbf{2}, \text{p. } 57)$$

$$\varphi = [1;\ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \cdots]. \qquad (\mathbf{8}, \text{p. } 697)$$

Because the partial quotients are repeating infinitely for the golden ratio, this continued fraction is called “periodic” and may be written

$$\varphi = [1;\ \bar{1}].$$

Some object to the use of continued fractions due to the labor involved in simplifying them. This objection carried more weight before the advent of modern technology which allows for quicker and more automatic simplification. Using a calculator or computer program two by two matrices can be multiplied to simplify the continued fraction.

Let matrix $\mathbf{A_i}$ represent each partial quotient in the continued fraction such that

$$\mathbf{A_i} = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, letting *n* equal the number of partial quotients in the finite continued fraction set

$$\mathbf{M} = \begin{pmatrix} p_n & c \\ q_n & d \end{pmatrix} = \mathbf{A_0 A_1 A_2} \cdots \mathbf{A_n}.$$

Short proves that the simplified form of the continued fraction will equal

$$\frac{p_n}{q_n}. \qquad (\mathbf{4}, \text{p. } 130)$$

Using matrices on a graphing calculator with successive partial quotients, the following best rational approximants are found for pi, *e*, and the golden ratio:

$$\pi \approx \left\{ \frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \ \cdots \right\},$$

$$e \approx \left\{ \frac{2}{1}, \frac{3}{1}, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \frac{106}{39}, \frac{193}{71}, \frac{1264}{465} \ \cdots \right\},$$

$$\varphi \approx \left\{ \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \ \cdots \right\}.$$

Chapter 4: More on Periodic Continued Fractions

In the previous chapter, the continued fraction representation of the golden ratio was given as a periodic continued fraction:

$$\varphi = [1;\ \overline{1}].$$

The more familiar definition of the golden ratio is

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

This type of number is called a *quadratic surd*. Quadratic surds are solutions of quadratic equations and are any number of the form

$$\frac{A+B\sqrt{C}}{D}, \tag{2}$$

where A, B, C and D are each integers, C is nonnegative, and D is not equal to zero.

Euler proved that the continued fraction form of any quadratic surd will always be periodic. Lagrange then proved the converse; that every periodic continued fraction will simplify to a quadratic surd (**10**, p. 207).

There is a distinction between continued fractions that are *purely periodic* and those which are *ultimately periodic*. With purely periodic continued fractions all of the partial quotients are repeated as with these examples:

$$\frac{1+\sqrt{5}}{2}=[1;\ \overline{1}],$$
$$\frac{1+\sqrt{2}}{2}=[2;\ \overline{2}],$$
$$\frac{5+3\sqrt{5}}{2}=[5;\ \overline{1,\ 5}],$$

and

$$\frac{4+\sqrt{37}}{7}=[1;\ \overline{2,\ 3,\ 1}].$$

More common are the ultimately periodic continued fractions where only a final subset of the partial quotients repeat. For example,

$$\frac{1+\sqrt{2}}{3}=[0;\ 1,\ \overline{4,\ 8}],$$
$$\sqrt{2}=[1;\ \overline{2}],$$

and

$$\sqrt{211}=[14;\ \overline{1,\ 1,\ 9,\ 5,\ 1,\ 2,\ 2,\ 1,\ 1,\ 4,\ 3,\ 1,\ 13,\ 1,\ 3,\ 4,\ 1,\ 1,\ 2,\ 2,\ 1,\ 5,\ 9,\ 1,\ 1,\ 28}]$$

are each ultimately periodic in that not all of their partial quotients repeat (**1**, p. 86).

Finally, it should be noted that terminating continued fractions may be considered ultimately periodic with the final partial quotient, zero, repeating. These special cases would represent the quadratic surd from (2) with

$$C = 0,$$

and are the rational roots of quadratic equations.

Chapter 5: Conclusion and Suggestions for the Curriculum

Although the fundamental theory of continued fractions discussed in this paper has been known for centuries, it is seldom taught before graduate school. Nevertheless, applications of the theory abound and the list continues to grow.

Mansfield gives a list of eighteen areas of applied mathematics which use continued fractions which he believes can be effectively taught in a two-year college. This list includes solving electrical network problems, approximating gear ratios, and making chemical and physical mixtures, in addition to numerous applications in number theory (**9**, p. 150).

Even though the author has an advanced degree in mathematics, and has taught public school mathematics for ten years, his first exposure to continued fractions was in a topology class in graduate school. In that class, the professor showed how continued fractions are fundamental in the study of knot theory. But the part that motivated the author to study more on the topic of continued fractions occurred when the professor showed how knot theory is currently being used by biologists in studying the behavior of viruses. In the search for a life saving cure to all types of viral diseases, knot theory, and thus the theory of continued fractions, is employed. It is the author's belief that such applications are so relevant that an exposure to them, and to the mathematics behind them, will motivate students to learn common fractions.

Students in middle school learn how to perform operations with and simplify fractions. In seventh grade, students plot points in all four quadrants of the coordinate plane. This would be a perfect time to use the $\mathbb{Z}^2$ lattice to model rational numbers. The author has done this with his sixth grade students with positive rational numbers. Representing rational numbers as points in the coordinate plane can also be used to teach addition and subtraction of fractions with unlike denominators. While teaching any of these topics, an extension into the simplification of simple continued fractions would be wholly appropriate, practical, and a perfect opportunity to expose students to some of the modern applications mentioned previously.

The traditional definition of a rational number is a real number that can be written as the ratio of two integers. The corresponding definition of an irrational number is a real number that cannot be written as the ratio of two integers. This definition is woefully inadequate. The following definition is more intuitive and visual.

> **Definition:** Let α be a point in the coordinate plane with abscissa α_x and ordinate α_y. If the line containing α and the origin does not contain any other point in the $\mathbb{Z}^2$ lattice, then the ratio of α_x and α_y will be irrational.

Starting with this definition, the construction of outpoints to approximate α leads directly to the definition of irrational numbers using continued fractions which was used in chapter two; namely, that an irrational number is one represented by an infinite continued fraction.

Finally, the discussion of periodic continued fractions removes a misconception which some students develop about irrational numbers. From the decimal form of irrational numbers, students can develop the idea that irrational numbers are chaotic and that their terms are random. By noting the fact that quadratic surds, while irrational, do indeed follow a repeating pattern in their continued fraction form, this misconception is eliminated.

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Vita

Baron Hannsz earned his BA in Mathematics at Texas A&M University in College Station, TX. He has taught mathematics for ten years; five in high school and five in middle school. His passion is making mathematics relevant to the average student and is always looking for new ways to present old material. He currently serves as a master teacher at New Caney High School in East Montgomery County, Texas, where he mentors and trains career teachers in best-practice teaching strategies.

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